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Einstein metrics: homogeneous solvmanifolds, generalised Heisenberg groups and black holes

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Abstract

In this paper we construct Einstein spaces with negative Ricci curvature in various dimensions. These spaces—which can be thought of as generalised AdS spacetimes—can be classified in terms of the geometry of the horospheres in Poincaré-like coordinates, and can be both homogeneous and static. By using simple building blocks, which in general are homogeneous Einstein solvmanifolds, we give a general algorithm for constructing Einstein metrics where the horospheres are products of generalised Heisenberg geometries, nilgeometries, solvegeometries, or Ricci-flat manifolds. Furthermore, we show that all of these spaces can give rise to black holes with the horizon geometry corresponding to the geometry of the horospheres, by explicitly deriving their metrics.

JGP SC: Riemannian geometry; General relativity

Keywords: Einstein solvmanifolds; Anti-de Sitter spaces (AdS); Black holes

1. Introduction

In the recent years the study of Anti-de Sitter spaces (AdS) has been intense. Because they arise as maximally symmetric solutions to the Einstein equations with a negative cosmological constant they were thought for a long time to be irrelevant to physics and merely a mathematical curiosity. However, from a mathematical point of view, negatively curved spaces have an incredibly rich structure [1–3]. For example, one of the biggest problems in classifying three-manifolds is the enormous number of compact hyperbolic spaces; in general, one finds the negatively curved spaces have an enormously diverse

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variety. For the AdS spaces—which can be thought of as the Lorentzian versions of the hyperbolic spaces—this diversity can, for example, be seen in the many AdS black holes that one knows of [4–8].

AdS spaces have also come in the focus of research after the advent of superstring theory [9]. One of the most promising ideas is the AdS/CFT correspondence [10] which relates a supergravity theory in the interior of AdS to a field theory in the boundary of AdS space. This correspondence illustrates the interplay between the structure of the interior of these spaces with the structure on the conformal boundary of the AdS space.

In this paper, we shall construct negatively curved Einstein manifolds which are, unlike the AdS spaces, not maximally symmetric. However, like the AdS spaces, a large class of them are homogeneous and static, though in general they need not be. However, there will always exist a non-trivial group acting on the space. This group manifests itself in horospherical coordinates where the space is foliated into horospheres. The Euclidean AdS spaces (or real hyperbolic spaces if one likes) in horospherical coordinates[1] are simply

$$\mathrm{d}s^2 = \mathbf{dw}^2 + \mathrm{e}^{-2w} \sum_{i=1}^{n} (\mathbf{dx}^i)^2. \tag{1}$$

Here, the horospheres are given by $w =$ constant and are flat Euclidean spaces. In this paper, we shall construct spaces for which the geometries of the horospheres are products of generalised Heisenberg groups, nilgeometries or solvegeometries. They are negatively curved Einstein spaces of the form

$$\mathrm{d}s^2 = \mathbf{dw}^2 + \sum_{i=1}^{n} \mathrm{e}^{-2q_i w} (\boldsymbol{\omega}^i)^2. \tag{2}$$

These spaces are analogous to the AdS spaces and their Lorentzian versions exist in any dimension higher than four. Higher-dimensional gravity has already given us some surprises, for example the great variety of black hole spacetimes (see [11] for a review) and even black strings [12]. In this paper we shall add even more black hole solutions to the myriad of known black hole spacetimes by finding black holes with horizons modelled on an arbitrary product of generalised Heisenberg groups, nilgeometries and solvegeometries. These black holes are not asymptotically AdS; they are asymptotically of the form (2).

This paper is organised as follows. First we consider in detail complex hyperbolic spaces. These spaces are the simplest non-maximally symmetric, negatively curved Einstein spaces. In the analysis, we discuss in detail how the horospheres can be equipped with a Heisenberg geometry. Then, in Section 3, we provide the simplest non-trivial example and solve the field equations for a space where the horospheres are a product between a Ricci-flat manifold and a Heisenberg geometry. The solution found is Einstein, and we discuss its isometries and give conditions for when the space is homogeneous. By considering a hypersurface in a product of Einstein spaces, we show in Section 4 how we can iteratively construct higher-dimensional Einstein spaces by using simple building blocks. These building blocks can be any manifold of a certain form, and in particular, they can be any homogeneous Einstein solvmanifold. Examples of such homogeneous Einstein solvmanifolds are given

[1] These are sometimes also called Poincaré coordinates.

in Section 5. Among these spaces are the so-called Damek–Ricci spaces which have horospheres equipped with generalised Heisenberg geometries. Also, examples where the horospheres are nilgeometries and solvegeometries are given. Lastly, we show that all spaces constructed by the iterative procedure have black hole analogues. We give explicit metrics for those black hole solutions which can have horizon geometries modelled on any product of generalised Heisenberg groups, nilgeometries and solvegeometries, and a Ricci-flat space.

2. Complex hyperbolic spaces

The metrics we are going to construct have many common features with the complex hyperbolic spaces, $\mathbb{H}_{\mathbb{C}}^{n+1}$. In fact, the construction is motivated from the existence and properties of these spaces. We will first review some of the aspects of the complex hyperbolic spaces which heavily motivated our construction. In this respect, the book by Goldman [13] is an indispensable source and reference.

The metric can be written in real horospherical coordinates

$$\mathrm{d}s^2 = \mathbf{dw}^2 + \mathrm{e}^{-2w}\left[\mathbf{dx} + \frac{1}{2}\sum_{k=1}^{n}(y^k\mathbf{dz}^k - z^k\mathbf{dy}^k)\right]^2 + \mathrm{e}^{-w}\sum_{k=1}^{n}[(\mathbf{dy}^k)^2 + (\mathbf{dz}^k)^2]. \tag{3}$$

The metric is a Kähler metric with constant holomorphic curvature, and thus, is also Einstein. This form of the metric is particularly useful for our purposes as we shall see. The full isometry group of this metric is $PU(n+1,1)$:

$$PU(n+1,1) \equiv \frac{U(n+1,1)}{\sim}, \quad \text{where } \mathsf{A} \sim \mathsf{B} \Leftrightarrow \mathsf{A} = \lambda\mathsf{B}, \;\; \lambda \in \mathbb{C}.$$

At the Lie algebra level this isometry group has the following Iwasawa decomposition:

$$\mathfrak{g} = \bigoplus_{k=-2}^{2} \mathfrak{g}_k, \tag{4}$$

and

$$\mathfrak{n}_\pm \equiv \mathfrak{g}_{\pm1} \oplus \mathfrak{g}_{\pm2}, \tag{5}$$

defines two copies of the $(2n+1)$-dimensional Heisenberg algebra. More precisely

$$\mathfrak{g}_0 \cong \mathfrak{u}(n) \times \mathbb{R}, \qquad \mathfrak{g}_{\pm1} \cong \mathbb{C}^n, \qquad \mathfrak{g}_{\pm2} \cong \mathbb{R}, \tag{6}$$

and $[\mathfrak{g}_i, \mathfrak{g}_j] = \mathfrak{g}_{i+j}$. In horospherical coordinates this decomposition is easy to interpret; the Heisenberg algebra $\mathfrak{n}_-$ generates a Heisenberg group,[2] $\mathfrak{H}$, which acts simply transitively on the horospheres $w =$ constant. Hence, each horosphere has a natural associated Heisenberg

[2] We will use a notation where lower case Gothic letters correspond to the Lie algebra; upper case Gothic to the Lie group; and calligraphic letters to the corresponding geometry (i.e. the Lie group equipped with a left-invariant metric).

geometry, $\mathcal{H}$. Furthermore, $\mathfrak{g}_0$ can be given a geometric interpretation in terms of this Heisenberg geometry as well. $\mathfrak{g}_0$ generates $U(n)$-rotations, and an $\mathbb{R}_+$-dilation with respect to the origin of $\mathcal{H}$. More explicitly, introducing the complex column vector

$$\boldsymbol{\zeta} = [y^1 + iz^1, y^2 + iz^2 \cdots y^n + iz^n]^{\mathrm{T}}, \tag{7}$$

the group $U(n)$ acts by matrix multiplication

$$\boldsymbol{\zeta} \overset{\mathsf{A}}{\mapsto} \mathsf{A}\boldsymbol{\zeta}. \tag{8}$$

The isometry group of $\mathcal{H}$ is now given by the semi-direct product

$$\mathrm{Isom}(\mathcal{H}) = U(n) \ltimes \mathfrak{H}. \tag{9}$$

On the Heisenberg space with coordinates $(x, \boldsymbol{\zeta})$ the dilation, ϕ_λ, acts as

$$(x, \boldsymbol{\zeta}) \overset{\phi_\lambda}{\mapsto} (\mathrm{e}^{2\lambda}x, \mathrm{e}^{\lambda}\boldsymbol{\zeta}),$$

which translates into an isometry of the metric (3) by acting along the w-coordinate:

$$(w, x, \boldsymbol{\zeta}) \overset{\phi_\lambda}{\mapsto} (w + \lambda, \mathrm{e}^{2\lambda}x, \mathrm{e}^{\lambda}\boldsymbol{\zeta}).$$

The *similarity group* of $\mathcal{H}$ is therefore the transformations generated by $\mathfrak{g}_0$ and the isometries $\mathfrak{H}$:

$$\mathrm{Sim}(\mathcal{H}) = (\mathbb{R}_+ \times U(n)) \ltimes \mathfrak{H}. \tag{10}$$

Note that the curve defined by $(x, \boldsymbol{\zeta}) =$ constant, is a geodesic from $w = -\infty$ to ∞. The geodesic connects two point on the boundary of $\mathbb{H}_{\mathbb{C}}^{n+1}$, denoted $\partial\mathbb{H}_{\mathbb{C}}^{n+1}$. The point at infinity, given by $w = \infty$, corresponds to one point on the boundary. We will call this point p_∞. The dilation maps the geodesic through the origin of the Heisenberg geometry onto itself. The algebra, $\mathfrak{g}_0$, generates all the isometries leaving the geodesic through the origin invariant.

Interestingly, the isometries induce a similarity group on the boundary of the complex hyperbolic space in the following sense. The boundary of $\mathbb{H}_{\mathbb{C}}^{n+1}$ is topologically a sphere S^{2n+1}. However, the isometries generated by $\mathfrak{g}_0$ and $\mathfrak{H}$, induce—via relation (10)—similarity transformations on the boundary minus the point at infinity: $\partial\mathbb{H}_{\mathbb{C}}^{n+1} - \{p_\infty\}$. The boundary $\partial\mathbb{H}_{\mathbb{C}}^{n+1} - \{p_\infty\}$ can therefore be considered as a Heisenberg geometry on which $\mathfrak{g}_0$ and $\mathfrak{H}$ acts as similarity transformations. In fact, the remaining isometries of $\mathbb{H}_{\mathbb{C}}^{n+1}$ will act on the boundary as conformal transformations. However, in our construction later on the remaining isometries will be broken; we will squash the complex hyperbolic space slightly along the w-coordinate.

This feature of complex hyperbolic spaces is very similar to the real hyperbolic space. In the real case, the horospheres have a Euclidean geometry, $\mathbb{E}^n$, and the similarity group $\mathrm{Sim}(\mathcal{H})$ is interchanged with the similarity group $\mathrm{Sim}(\mathbb{E}^n)$. This is also one of the many reasons why the AdS spaces are so interesting: they are the Lorentzian versions of real hyperbolic spaces. It is therefore interesting to know whether it is possible to extend the interesting geometric properties of the complex hyperbolic spaces to the Lorentzian case.

3. Some Einstein metrics of dimension $(2+2n+m)$

By including extra dimensions it is possible to construct spaces for which the structure (10) remains intact. The starting point is a squashed complex hyperbolic space but where we keep the some of the properties which we discussed earlier. We construct the Einstein metrics as follows: assume that $\mathcal{M}$ is an m-dimensional Ricci-flat manifold with metric $\widetilde{\mathrm{d}s}_m^2$, i.e.

$$\widetilde{\mathrm{d}s}_m^2 = \tilde{g}_{AB}\mathbf{d}\chi^A\mathbf{d}\chi^B, \qquad \tilde{R}_{AB} = 0. \tag{11}$$

Then we start with the metric ansatz where the above metric is warped in the following way[3]

$$\mathrm{d}s^2 = \mathrm{e}^{-2pw}\widetilde{\mathrm{d}s}_m^2 + Q^2\mathbf{dw}^2 + \mathrm{e}^{-2w}\left[\mathbf{dx} + \frac{1}{2}\sum_{k=1}^{n}(y^k\mathbf{dz}^k - z^k\mathbf{dy}^k)\right]^2 + \mathrm{e}^{-w}\sum_{k=1}^{n}[(\mathbf{dy}^k)^2 + (\mathbf{dz}^k)^2], \tag{12}$$

where Q and p are constants.

The metric ansatz immediately leads to a diagonal Ricci tensor (see Appendix A) and the large number of symmetries makes the field equations particularly easy to solve. For the values

$$p = \frac{n+2}{2(n+1)}, \qquad Q^2 = 1 + \frac{m(n+2)}{2(n+1)^2}, \tag{13}$$

the metric (12) is an Einstein metric with

$$R_{\mu\nu} = -\frac{n+2}{2}g_{\mu\nu}. \tag{14}$$

Of course, by rescaling the metric (12), we can find a metric with $R_{\mu\nu} = -\alpha^2 g_{\mu\nu}$ for any $\alpha^2 > 0$. Thus, in the following we will consider one particular α^2, but bear in mind that any rescaling is possible.

These spaces have many similarities with the AdS spaces and some of their properties are directly related to the complex hyperbolic space in horospherical coordinates. The isometry group of (12) depends on the properties of $\mathcal{M}$, and in general the metric (12) will not be homogeneous. Note that if there exists a proper similarity transformation of $\widetilde{\mathrm{d}s}_m^2$

$$\psi_\kappa : \chi^A \mapsto \psi_\kappa(\chi^A), \qquad \psi_\kappa^*\widetilde{\mathrm{d}s}_m^2 = \mathrm{e}^{2\kappa}\widetilde{\mathrm{d}s}_m^2, \tag{15}$$

where κ is a non-zero constant, then the dilation, ϕ_λ, can be extended to an isometry of (12):

$$(\chi^A, x, \boldsymbol{\zeta}) \overset{\phi_\lambda}{\mapsto} (\psi_{p\lambda}(\chi^A), \mathrm{e}^{2\lambda}x, \mathrm{e}^{\lambda}\boldsymbol{\zeta}). \tag{16}$$

[3] Later, we shall give a different interpretation of this metric; namely as a hypersurface in a higher-dimensional space.

This leads to the following observation: *if $\mathcal{M}$ is a homogeneous space and, in addition, allows for a proper homothety, then the metric given by* Eq. (12) *is homogeneous*.

For the metric (12) to be homogeneous, the existence of a proper homothety of $\mathcal{M}$ is crucial; however, there are still quite a few spaces having these properties. For example, the following Ricci-flat spaces are homogeneous and possess a proper homothety:

1. The Euclidean spaces $\mathbb{E}^m$, or the Minkowski spaces $\mathbb{M}^m$.
2. The m-dimensional Milne universes.
3. The m-dimensional homogeneous vacuum plane-waves [14].[4]

There are also examples of homogeneous spacetimes $\mathcal{M}$ which do not admit a proper homothety, and inhomogeneous spacetimes admitting a proper homothety.[5]

4. Extending the scope: "prime decomposition"

We will here describe an iterative procedure of constructing generalisations of the metric (12). It is based on a simple observation regarding the Gauss' equation for hypersurfaces.

Assume we have two negatively curved Einstein manifolds, $\mathcal{M}$, and $\mathcal{N}$, with metrics of the form

$$\mathrm{d}s^2_{\mathcal{M}} = \mathbf{dw}^2 + \sum_{i=1}^{m} \mathrm{e}^{-2p_i w}(\boldsymbol{\omega}^i)^2, \qquad R_{AB} = -\alpha^2 g_{AB}, \tag{17}$$

$$\mathrm{d}s^2_{\mathcal{N}} = \mathbf{dv}^2 + \sum_{i=1}^{n} \mathrm{e}^{-2q_i v}(\boldsymbol{\chi}^i)^2, \qquad R_{ab} = -\beta^2 g_{ab}. \tag{18}$$

It is essential here that the one-forms $\boldsymbol{\omega}^i$ and $\boldsymbol{\chi}^i$ both form a closed algebra, i.e. all the w and v dependence is in the exponential prefactor. Explicitly, this means that the forms $\boldsymbol{\omega}^i$ obey

$$\mathbf{d}\boldsymbol{\omega}^i = -\boldsymbol{\Omega}^i_{\ j} \wedge \boldsymbol{\omega}^j, \tag{19}$$

and $\boldsymbol{\Omega}^i_{\ j}$ does not involve w.

The construction now goes as follows. We define the product space $\mathcal{M} \times \mathcal{N}$ and consider a hypersurface $\Sigma \subset \mathcal{M} \times \mathcal{N}$. Our aim is to tune the parameters and choose the hypersurface such that the induced metric, $h_{\mu\nu}$, on Σ is an Einstein metric.

From the theory of hypersurfaces, we have the well-known result (the contracted Gauss' equation, see e.g. [16])

$${}^{(\Sigma)}R_{\mu\nu} = {}^{(\mathcal{M}\times\mathcal{N})}R_{\alpha\beta}h^{\alpha}_{\mu}h^{\beta}_{\nu} - {}^{(\mathcal{M}\times\mathcal{N})}R^{\lambda}_{\ \alpha\sigma\beta}n_{\lambda}n^{\sigma}h^{\alpha}_{\mu}h^{\beta}_{\nu} + KK_{\mu\nu} - K^{\alpha}_{\mu}K_{\alpha\nu}. \tag{20}$$

Here, n^{μ} and $K_{\mu\nu}$ are the normal vector field and the extrinsic curvature to the hypersurfaces, respectively.

[4] See also [15] which has some interesting negatively curved Einstein spaces conformally equivalent to the vacuum plane-waves.

[5] See also [11] where examples of compact Ricci-flat manifolds are provided.

Now using the metrics (17) and (18), the Ricci tensor in $\mathcal{M}\times\mathcal{N}$ becomes block diagonal. Let us consider the following hypersurface

$$v=\gamma w, \tag{21}$$

where γ is a constant. The unit normal is now given by

$$\mathbf{n}=-\frac{\gamma}{\sqrt{1+\gamma^2}}\frac{\partial}{\partial w}+\frac{1}{\sqrt{1+\gamma^2}}\frac{\partial}{\partial v}.$$

By calculating the Riemann tensor for $\mathcal{M}\times\mathcal{N}$ we get the following for the metrics given in Eqs. (17) and (18)

$${}^{(\mathcal{M}\times\mathcal{N})}R^{\lambda}_{\alpha\sigma\beta}n_\lambda n^\sigma h^\alpha_\mu h^\beta_\nu=-K^\alpha_\mu K_{\alpha\nu}.$$

Hence, Gauss' equation simplifies to

$${}^{(\Sigma)}R_{\mu\nu}={}^{(\mathcal{M}\times\mathcal{N})}R_{\alpha\beta}h^\alpha_\mu h^\beta_\nu+KK_{\mu\nu}. \tag{22}$$

Note there are two simple choices for making Σ an Einstein space. In both cases we choose $\alpha^2=\beta^2$ (by rescaling the metric). The first case arises when $K_{\mu\nu}\propto h_{\mu\nu}$. However, as can be seen, this happens only when $p_i=q_j\equiv p$. This is a well-known example and does not give any new solutions. The other case is more interesting and is defined by the choice $K=0$. So, the following will lead to an Einstein space:

$$\alpha^2=\beta^2,\qquad K=0. \tag{23}$$

Explicitly, the requirement $K=0$ leads to

$$\gamma\sum_{i=1}^m p_i-\sum_{i=1}^n q_i=0. \tag{24}$$

From this, γ can be found and the metric on Σ becomes (after rescaling w)

$$\mathrm{d}s^2_\Sigma=\mathbf{dw}^2+\sum_{i=1}^m \mathrm{e}^{-2P_iw}(\boldsymbol{\omega}^i)^2+\sum_{i=1}^n \mathrm{e}^{-2Q_iw}(\boldsymbol{\chi}^i)^2,$$

$$P_i=\frac{1}{\sqrt{1+\gamma^2}}p_i,\qquad Q_i=\frac{\gamma}{\sqrt{1+\gamma^2}}q_i. \tag{25}$$

Using this procedure, we can iteratively construct negatively curved Einstein spaces in higher dimensions using building blocks of the form (17). Thus we have provided a product rule for the geometry of the horosphere. Assume that $M(\mathcal{H}_1)$ and $M(\mathcal{H}_2)$ are Einstein metrics of the above form with horospheres equipped with the geometries $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Then we have a product rule

$$M(\mathcal{H}_1)\odot M(\mathcal{H}_2)=M(\mathcal{H}_1\times\mathcal{H}_2), \tag{26}$$

given by $M(\mathcal{H}_1)\odot M(\mathcal{H}_2)\cong\Sigma\subset M(\mathcal{H}_1)\times M(\mathcal{H}_2)$ with the induced metric. Note that

$$[M(\mathcal{H}_1)\odot M(\mathcal{H}_2)]\odot M(\mathcal{H}_3)=M(\mathcal{H}_1)\odot[M(\mathcal{H}_2)\odot M(\mathcal{H}_3)],$$

i.e. the operation $\odot$ is associative. In this way the classification reduces to considering irreducible "prime" manifolds $M(\mathcal{H}_i)$.

4.1. General formula

In fact, we can also give a general formula for the exponents. Assume that we have N Einstein spaces

$$\mathrm{d}s_A^2 = \mathbf{dw}^2 + \sum_{i=1}^{m_A} \mathrm{e}^{-2p_{i(A)}w}(\boldsymbol{\omega}^i_{(A)})^2, \qquad R_{\mu\nu} = -\alpha^2 g_{\mu\nu}, \quad A = 1, \dots, N. \tag{27}$$

Then there is an Einstein space

$$\mathrm{d}s^2 = \mathbf{dw}^2 + \sum_{A=1}^{N}\sum_{i=1}^{m_A} \mathrm{e}^{-2q_{i(A)}w}(\boldsymbol{\omega}^i_{(A)})^2, \qquad R_{\mu\nu} = -\alpha^2 g_{\mu\nu}, \tag{28}$$

where

$$q_{i(A)} = p_{i(A)}\left(\sum_{j=1}^{m_A} p_{j(A)}\right)\left[\sum_{B=1}^{N}\left(\sum_{j=1}^{m_B} p_{j(B)}\right)^2\right]^{-1/2}. \tag{29}$$

5. Einstein solvmanifolds

Note that we have assumed in the above construction that the metric can be more general than the usual complex and real hyperbolic spaces. Thus we may wonder if there are other “building blocks” than the ones already considered of the form (17). Indeed there are and, in fact, they are so numerous that not all such manifolds are known. However, here we will consider some generalisations of these building blocks; namely homogeneous Einstein solvmanifolds.

Let us consider the homogeneous spaces for which the Lie algebras, $\mathfrak{s}$, obey (see e.g. [17]):

1. The Iwasawa decomposition has the following orthogonal decomposition:

   $$\mathfrak{s} = \mathfrak{a} \oplus \mathfrak{n}, \quad [\mathfrak{s}, \mathfrak{s}] = \mathfrak{n},$$

   where $\mathfrak{a}$ is abelian, and $\mathfrak{n}$ is nilpotent.
2. All operators ad_X, $X \in \mathfrak{a}$ are symmetric.
3. For some $X^0 \in \mathfrak{a}$, $\mathrm{ad}_{X^0}|_{\mathfrak{n}}$ has positive eigenvalues.

These Lie algebras give rise to group manifolds, and since $\mathfrak{s}$ is a solvable algebra, these are so-called *solvmanifolds*. By using the left-invariant one-forms we can turn the groups into Riemannian homogeneous spaces. These spaces can always be put onto the form (17) and they are strong candidates for Einstein spaces with negative curvature [17,18].

The different classes of these solvmanifolds are characterised by the dimension of $\mathfrak{a}$ and the property of the nilpotent part $\mathfrak{n}$. Some general results are known, but a complete list of manifolds of this type is lacking. Here, due to the multiplication rule as given above, we will only consider the cases where $\mathfrak{s}$ leads to an irreducible solvmanifold $M(\mathcal{H})$.

5.1. $dim(\mathfrak{a}) = 1$, *and* $\mathfrak{n}$ *abelian*

These are the well-known hyperbolic spaces. The metric can be written on the form given in Eq. (1).

5.2. $dim(\mathfrak{a}) = 1$, *and* $\mathfrak{n}$ *generalised Heisenberg algebra: Damek–Ricci spaces*

In this case, we can construct all possible cases. The nilpotent part $\mathfrak{n}$ are generalised Heisenberg algebras, and the extended solvable group give rise to so-called Damek–Ricci spaces.

5.2.1. *Generalised Heisenberg algebras*

The (ordinary) $(2n + 1)$-dimensional Heisenberg algebra is the nilpotent part of the Iwasawa decomposition of $\mathrm{Isom}(\mathbb{H}_{\mathbb{C}}^{n+1})$. Here we will consider the *generalised Heisenberg spaces*,[6] $\mathcal{H}_{m,n}$. The generalised Heisenberg algebras are defined as follows. Let $\mathfrak{b}$ and $\mathfrak{z}$ be real vector spaces of dimension m and n, respectively, such that $\mathfrak{n}$ is the orthogonal sum $\mathfrak{n} = \mathfrak{b} \oplus \mathfrak{z}$. Assume that the commutator is a map $[-, -] : \mathfrak{n} \times \mathfrak{n} \mapsto \mathfrak{z}$ and $[\mathfrak{z}, \mathfrak{z}] = 0$. Furthermore, assume that there exists a $J : \mathfrak{z} \mapsto \mathrm{End}(\mathfrak{b})$, $Z \mapsto J_Z$ such that

$$\langle [X, Y], Z\rangle = \langle Y, J_Z X\rangle, \quad X, Y \in \mathfrak{b}, \quad Z \in \mathfrak{z}.$$

The triple $(\mathfrak{b}, \mathfrak{z}, J)$ defines uniquely a two-step nilpotent algebra $\mathfrak{n}$ and, in fact, a two-step nilpotent simply connected Lie group with a left-invariant metric. We say that $\mathfrak{n}$ is a *generalised Heisenberg algebra* if, in addition

$$J_Z^2 = -\langle Z, Z\rangle \mathrm{Id}_{\mathfrak{b}}, \quad \forall Z \in \mathfrak{z}.$$

The generalised Heisenberg spaces, $\mathcal{H}_{m,n}$, are the corresponding group manifolds equipped with orthonormal left-invariant one-forms of the form

$$\boldsymbol{\omega}^A = \mathbf{dx}^A + \tfrac{1}{2} B^A_{ab} y^a \mathbf{dy}^b, \quad A = 1, \dots, m, \qquad \boldsymbol{\omega}^a = \mathbf{dy}^a, \quad a = 1, \dots, n. \tag{30}$$

Here are B^A_{ab} antisymmetric in the lower indices. It should be noted that, given an m, not all n's are allowed. For example, for $m = 1$, which gives the ordinary Heisenberg spaces, only even n are allowed. In general, the number n can have the following values:

$$n = kn_0, \quad k \in \mathbb{N}, \tag{31}$$

where n_0 is given in Table 1. For each of these values there is a unique Heisenberg space, except for $m = 3 \pmod 4$ where there can be many non-equivalent Heisenberg spaces for a given dimension (see [19] for details).

In particular, the generalised Heisenberg algebras are two-step nilpotent, i.e. $[[\mathfrak{n}, \mathfrak{n}], \mathfrak{n}] = 0$. As the Lie exponential map, $\exp_{\mathfrak{n}} : \mathfrak{n} \mapsto \mathfrak{H}_{m,n}$, is a diffeomorphism, the Heisenberg group, $\mathfrak{H}_{m,n}$, will also be a two-step nilpotent group. However, note that not all two-step nilpotent groups are generalised Heisenberg groups.[7]

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6 See e.g. [19].

7 For example, the two-step nilpotent algebra $A_{5,1}$ in [31] is not a generalised Heisenberg algebra.

Table 1
Generalised Heisenberg spaces: the different allowed values of n_0 for a given m

m	$8p$	$8p+1$	$8p+2$	$8p+3$	$8p+4$	$8p+5$	$8p+6$	$8p+7$
n_0	2^{4p}	2^{4p+1}	2^{4p+2}	2^{4p+2}	2^{4p+3}	2^{4p+3}	2^{4p+3}	2^{4p+3}

5.2.2. *Damek–Ricci spaces*

Using these generalised Heisenberg groups we can construct other group manifolds which are called *Damek–Ricci spaces*. These spaces are group manifolds similar to the complex hyperbolic spaces and have an Iwasawa decomposition where the generalised Heisenberg algebras will appear [20]. The Damek–Ricci space $S_{m,n}$, is defined as the manifold having the (globally defined) orthonormal left-invariant one-forms [19]

$$\begin{aligned}&\boldsymbol{\omega}^A = \mathrm{e}^{-w}(\mathbf{dx}^A + \tfrac{1}{2}B^A_{ab}y^a\mathbf{dy}^b), \quad A = 1,\ldots,m, \qquad \boldsymbol{\omega}^a = \mathrm{e}^{-w/2}\mathbf{dy}^a,\\ &a = 1,\ldots,n, \qquad \boldsymbol{\omega}^w = \mathbf{dw}.\end{aligned} \tag{32}$$

Hence, they are of dimension $(n+m+1)$. Furthermore, the Damek–Ricci spaces are Einstein manifolds with negative curvature [21]:

$$R_{\mu\nu} = -(m + \tfrac{1}{4}n)g_{\mu\nu}. \tag{33}$$

Thus they are spaces obeying (17).

As explained earlier, these Damek–Ricci spaces are homogeneous group manifolds of solvable type. The derived Lie algebra, $[\mathfrak{s},\mathfrak{s}] = \mathfrak{n}$, is of generalised Heisenberg type.

Note that the case $m = 1$ we have $S_{1,2n} \cong \mathbb{H}^{n+1}_{\mathbb{C}}$, so these spaces generalise the complex hyperbolic spaces. Moreover, the quaternionic hyperbolic space $\mathbb{H}^{n+1}_{\mathbb{H}}$, is one of the $S_{3,4n}$ spaces, and the Cayley hyperbolic plane, $\mathsf{Cay}\mathbb{H}^2$, is isometric to the $S_{7,8}$ space. Hence, all of the division algebras can be realised in these spaces.[8]

5.3. *dim*$(\mathfrak{a}) = 1$, *and* $\mathfrak{n}$ *nilpotent*

If we allow for the $\mathfrak{n}$ to be a more general nilpotent algebra, many more possibilities arise. Unfortunately, not all nilpotent algebras are known; however, all nilpotent algebras up to dimension 7 are given in [25]. Other examples, such as some infinite series of nilpotent algebras, are given in [17].

For all these spaces, the metric can be written in horospherical coordinates where the horospheres are *nilgeometries*, Nil^n. As an example, we can consider the space having Nil^4 horospheres.[9] Let Nil^4 have the orthonormal left-invariant one-forms

$$\boldsymbol{\omega}^1 = \mathbf{dv}, \qquad \boldsymbol{\omega}^2 = \mathbf{dx} - y\mathbf{dv}, \qquad \boldsymbol{\omega}^3 = \mathbf{dy} - z\mathbf{dv}, \qquad \boldsymbol{\omega}^4 = \mathbf{dz}. \tag{34}$$

[8] For more about these spaces and their relatives, consult for example [17,22–24].

[9] There is only one irreducible four-dimensional real Lie algebra which is nilpotent. This algebra is three-step nilpotent and therefore it is not a generalised Heisenberg algebra.

Then the metric of the form (17) with $\boldsymbol{\omega}^i$ given as above, and

$$(p_1, p_2, p_3, p_4) = \left(\frac{1}{2\sqrt{5}}, \frac{2}{\sqrt{5}}, \frac{3}{2\sqrt{5}}, \frac{1}{\sqrt{5}}\right) \tag{35}$$

is an Einstein space with $R_{\mu\nu} = -(3/2)g_{\mu\nu}$.

5.4. *dim*$(\mathfrak{a}) > 1$, *and* $\mathfrak{n}$ *nilpotent*

In this case, the horospheres are solvmanifolds themselves. All of these cases are not known, but one can find some particular examples of such spaces.

For example, there are solutions as follows. Consider the solvegeometry having the orthonormal left-invariant one-forms

$$\boldsymbol{\omega}^n = \mathbf{dv}, \qquad \boldsymbol{\omega}^i = \mathrm{e}^{-q_i v}\mathbf{dx}^i, \quad i = 1, \dots, (n-1), \qquad \sum_{i=1}^{n-1} q_i = 0. \tag{36}$$

Then the metric of the form (17) with $\boldsymbol{\omega}^i$ given as above, and

$$p_1 = p_2 = \cdots = p_{n-1} \equiv p, \quad p_n = 0, \qquad p^2 = \frac{1}{n-1}\sum_{i=1}^{n-1} q_i^2 \tag{37}$$

is an Einstein space with $R_{\mu\nu} = -(n-1)p^2 g_{\mu\nu}$.

6. Black holes

As we now iteratively have constructed spaces for which the horospheres are products of generalised Heisenberg spaces, nilgeometries, solvegeometries and a Ricci-flat space, one might wonder if there are similar generalisations of black holes. Indeed there are.

Assume there is a negatively curved Einstein space given by

$$\mathrm{d}s^2 = -\mathrm{e}^{-2pw}\mathbf{dt}^2 + \mathbf{dw}^2 + \sum_{i=1}^{n} \mathrm{e}^{-2q_i w}(\boldsymbol{\omega}^i)^2, \qquad R_{\mu\nu} = -\alpha^2 g_{\mu\nu} \tag{38}$$

(here, p must be $p = (\sum q_i^2)/(\sum q_i)$). Then there is a corresponding "black hole" spacetime for which the metric takes the form

$$\mathrm{d}s^2 = -\mathrm{e}^{-2pw}F(w)\mathbf{dt}^2 + \frac{\mathbf{dw}^2}{F(w)} + \sum_{i=1}^{n} \mathrm{e}^{-2q_i w}(\boldsymbol{\omega}^i)^2,$$
$$F(w) = 1 - M\exp\left[\left(p + \sum_{i=1}^{n} q_i\right)w\right], \quad R_{\mu\nu} = -\alpha^2 g_{\mu\nu}. \tag{39}$$

Hence, by doing a proper identification we can from the above construct black hole spacetimes which have horizon geometries modelled on

$$\mathcal{S} \cong \mathcal{R}^m \times \mathcal{M}_1 \times \mathcal{M}_2 \times \cdots \times \mathcal{M}_k,$$

where $\mathcal{R}^m$ is an m-dimensional Ricci-flat manifold and each $\mathcal{M}_i$ is a generalised Heisenberg group, nilgeometry or a solvegeometry, as follows: we choose a discrete group $\Gamma \subset \mathrm{Isom}(\mathcal{S})$ which acts freely and properly discontinuously on $\mathcal{S}$, and construct the quotient $\mathcal{S}/\Gamma$. Locally, as $w \to -\infty$, all these black hole spacetimes asymptotically approach the spacetime given in Eq. (38).

Here a comment is required. One usually assume that the horizon of the black hole has finite volume (possibly also compact). Not all geometries of the above class allow for a finite volume quotient, i.e. that $\mathcal{S}/\Gamma$ has finite volume. Assuming a finite volume horizon, we have to restrict ourselves to model geometries in the sense of Thurston [1].

Model geometry: a pair (X, G) with X a connected and simply connected manifold, and G a Lie group acting transitively on X, is called a *model geometry* if the following conditions are satisfied:

1. X can be equipped with a G-invariant Riemannian metric.
2. G is maximal, i.e. there does not exist a larger group $H \supset G$, where H acts transitively on X and requirement 1 is satisfied.
3. There exists a discrete subgroup $\Gamma \subset G$ such that X/Γ has finite volume.

The model geometries in dimension 3 were found by Thurston [1,2] and are usually called “the 8 Thurston geometries”. The four-dimensional model geometries were found by Filipkiewicz [26] (see also [27,28]). Hence, by allowing only model geometries as the geometry of the horospheres, we ensure that there exists a $\Gamma \subset \mathrm{Isom}(\mathcal{S})$ such that $\mathcal{S}/\Gamma$ has finite volume.[10]

Note that the horizons of these black holes are not Einstein manifolds which implies that the analysis done in [11] is not applicable for these black holes. However, it would be interesting to do a similar analysis and check whether or not these black holes are stable.

6.1. Six-dimensional black holes modelled on solvegeometries

Let us consider a six-dimensional example, where the horizon is modelled on the infinite series of model geometries called[11] $\mathsf{Sol}^4_{m,n}$.

The solvable Lie groups $\mathsf{Sol}^4_{m,n}$ can be considered as $\mathsf{Sol}^4_{m,n} = \mathbb{R}^3 \ltimes_A \mathbb{R}$, where A is the matrix

$$A = \exp\begin{bmatrix} at & 0 & 0 \\ 0 & bt & 0 \\ 0 & 0 & ct \end{bmatrix}. \tag{40}$$

Here, $a > b > c$, $a + b + c = 0$, and $\lambda_i = \mathrm{e}^a, \mathrm{e}^b, \mathrm{e}^c$ are the roots of the cubic

$$\lambda^3 - m\lambda^2 + n\lambda - 1 = 0 \tag{41}$$

[10] The five-dimensional black hole solutions with horizon geometry modelled on the three-dimensional model geometries were considered in [29]. See also [30] in this respect.

[11] There are two other solvegeometries in dimension 4 which are model geometries; namely the ones called Sol^4_0 and Sol^4_1. In the notation of Patera et al. [31,32], Sol^4_1 has the simply transitive group $A_{4,8}$ while Sol^4_0 has $A^{-1/2,-1/2}_{4,5}$ and the infinite series $A^{-2q,q}_{4,6}$ [33].

with m, n positive integers. Note that if $m = n$, we have $\lambda_2 = 1$ and thus $\mathsf{Sol}^4_{m,m} = \mathsf{Sol}^3 \times \mathbb{E}^1$. Proportional matrices A have isomorphic geometries. An invariant metric is[12]

$$\mathrm{d}\sigma^2 = \mathbf{dv}^2 + \mathrm{e}^{av}\mathbf{dx}^2 + \mathrm{e}^{bv}\mathbf{dy}^2 + \mathrm{e}^{cv}\mathbf{dz}^2. \tag{42}$$

Hence, this metric is included in the example in Section 5.4. This means that by using this example, and the Einstein metric

$$\mathbf{dw}^2 - \mathrm{e}^{-2qw}\mathbf{dt}^2, \tag{43}$$

we can construct a six-dimensional Einstein space by

$$M(\mathbb{E}^1) \odot M(\mathsf{Sol}^4_{m,n}) = M(\mathbb{E}^1 \times \mathsf{Sol}^4_{m,n}). \tag{44}$$

By using the corresponding metric (39) we can construct a black hole metric where the horizon is modelled on the geometries $\mathsf{Sol}^4_{m,n}$.

This is only one example of the possible black holes one can construct in six dimensions. Using, for example, the example in Section 5.3 we can similarly construct six-dimensional black holes modelled on Nil^4.

7. Summary

In this paper we have systematically constructed negatively curved Einstein spaces of various dimensions. The spaces can be classified in terms of the geometry of the horospheres, and by using a set of building blocks we gave an iterative procedure of constructing higher-dimensional Einstein spaces for which the geometry of the horospheres was an arbitrary product of generalised Heisenberg spaces, nilgeometries, solvegeometries, and a Ricci-flat space. The building blocks could be any of the homogeneous Einstein solvmanifolds. We also showed that all of these spaces have black hole analogues by explicitly writing down the metrics. These black holes provide us with an infinite series of topologically distinct black holes in higher dimensions. The horizon of these black holes are not Einstein in general, however, the non-Einstein part is always modelled on the so-called model geometries.

Here in this paper, only a specific type of black holes were considered. However, the Einstein metrics also allow for BTZ black holes [4,8]. For example, using the metric (12) with $m = 1$, we can identify points under the action of $\phi_\lambda \circ \mathsf{A}$, where ϕ_λ is the dilation and $\mathsf{A} \in U(n-1)$, to create a black hole. By performing this identification we create BTZ black holes similar to those found by Bañados [34]. These BTZ constructions have not been explored in this paper, but it would certainly be interesting to investigate these BTZ analogues as well.

This paper has set the scene for investigation of higher-dimensional black holes with horizon geometries which are not Einstein. Several unanswered questions still remain. For example, are these black holes stable? What role does the boundary of these spaces play

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[12] This geometry corresponds to the Lie algebra $A^{p,q}_{4,5}$ with $p = b/a$ and $q = c/a$ in the notation of Patera et al. [31,32].

for the physics in the interior? Is there an AdS/CFT version for these spaces? So far, none of these questions have been investigated. Only time, and some more work, can tell what their answers might be.

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Appendix A. Curvature tensors

Consider the metric

$$\mathrm{d}s^2 = -\mathrm{e}^{-2pw}\,\mathrm{e}^{2\beta}\mathbf{dt}^2 + \mathrm{e}^{-2\beta}\mathbf{dw}^2 + \sum_{i=1}^{n}\mathrm{e}^{-2q_i w}(\boldsymbol{\omega}^i)^2, \tag{A.1}$$

where $\beta = \beta(w)$. Assume also that the metric with $\beta = 0$ is *homogeneous* where $\mathrm{e}^{-q_i w}\boldsymbol{\omega}^i$ are left-invariant one-forms, and that the space with metric

$$\widetilde{\mathrm{d}s}^2 = \sum_{i=1}^{n}(\boldsymbol{\omega}^i)^2 \tag{A.2}$$

is homogeneous and has the Riemann and Ricci tensors given by $\tilde{R}^i_{jkl}$ and $\tilde{R}_{ij}$, respectively. These spaces include most of the cases considered in this paper.

In the orthonormal frame, the independent components of the Riemann tensor are given by (no sum unless explicitly specified)

$$\begin{aligned} &R^t_{wtw} = -(\beta'' + 2(\beta')^2 - 3p\beta' + p^2)\,\mathrm{e}^{2\beta}, \qquad R^t_{iti} = -q_i(p-\beta')\,\mathrm{e}^{2\beta},\\ &R^i_{wiw} = -q_i(q_i - \beta')\,\mathrm{e}^{2\beta}, \qquad \underset{i\neq j}{R^i_{jij}} = \tilde{R}^i_{jij} - q_i q_j\,\mathrm{e}^{2\beta}, \qquad \text{rest of } \; R^i_{jkl} = \tilde{R}^i_{jkl}. \end{aligned} \tag{A.3}$$

The Ricci tensor is thus

$$\begin{aligned} R_{tt} &= \left[\beta'' + 2(\beta')^2 - 3p\beta' + p^2 + (p-\beta')\sum_{i=1}^{n} q_i\right]\mathrm{e}^{2\beta},\\ R_{ww} &= -\left[\beta'' + 2(\beta')^2 - 3p\beta' + p^2 + \sum_{i=1}^{n} q_i(q_i - \beta')\right]\mathrm{e}^{2\beta},\\ R_{ij} &= \tilde{R}_{ij} - \delta_{ij}q_i\left(\sum_{k=1}^{n} q_k + p - 2\beta'\right)\mathrm{e}^{2\beta}. \end{aligned} \tag{A.4}$$

For an Einstein space we have $R_{tt} + R_{ww} = 0$, which implies $p = (\sum q_i^2)/(\sum q_i)$. Note also that the non-black hole solutions are given by $\beta' = \beta = 0$.

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